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Positive solutions of fourth-order nonlinear singular Sturm–Liouville eigenvalue problems

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Abstract

We consider the existence of positive solutions for the following fourth-order singular Sturm–Liouville eigenvalue problems

$$\begin{cases} \dfrac{1}{p(t)}\big(p(t)u'''(t)\big)' - \lambda g(t)F(t,u,u'') = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 \lim_{t\to 0+} p(t)u'''(t) = 0, \\ \gamma_2 u''(1) + \delta_2 \lim_{t\to 1-} p(t)u'''(t) = 0, \end{cases}$$

where $\lambda > 0$, g, p may be singular at $t = 0$ and/or 1. Moreover, $F(t,x,y)$ may also have singularity at $x = 0$ and/or $y = 0$. By using fixed point theory in cones, an explicit interval for λ is derived such that for any λ in this interval, the existence of at least one positive solution to the boundary value problem is guaranteed. Our results extend and improve many known results including singular and nonsingular cases.

Keywords: Fourth-order singular differential equation; Eigenvalue problems; Positive solutions; Fixed point theory

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1. Introduction

In this paper, we will study the existence of positive solutions for the following fourth-order nonlinear singular Sturm–Liouville eigenvalue problems

$$\begin{cases} \frac{1}{p(t)}\big(p(t)u'''(t)\big)' - \lambda g(t)F(t,u,u'') = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 \lim_{t\to 0+} p(t)u'''(t) = 0, \\ \gamma_2 u''(1) + \delta_2 \lim_{t\to 1-} p(t)u'''(t) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\alpha_i, \beta_i, \delta_i, \gamma_i \geqslant 0$ $(i = 1, 2)$ are constants such that $\rho_i = \beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$ $(i = 1, 2)$, and $p \in C^1((0,1),(0,+\infty))$, $0 < \int_0^1 \frac{ds}{p(s)} < +\infty$. Moreover g, p may be singular at $t = 0$ and/or 1, and $F(t,x,y)$ may also have singularity at $x = 0$ and/or $y = 0$.

The boundary value problems for ordinary differential equations play a very important role in both theory and application. They are used to describe a large number of physical, biological and chemical phenomena. Equation (1.1) is often referred to as the deformation of an elastic beam under a variety of boundary conditions, for details, see [1–14]. For example, problem (1.1) subject to Lidstone boundary value conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ is used to model such phenomena as the deflection of elastic beam simply supported at the endpoints, see [1,3,5,7–11, 13,14]. In applications, we are interested in showing the existence of positive solutions for $\lambda > 0$. Particularly, when $p(t) = 1$, $\lambda g(t)F(t,u(t),u''(t)) = e(t) - f(t,u(t)) + \pi^4 u(t)$, where $f(t,u)$ is strictly increasing on u for every $t \in [0,1]$ and $\int_0^1 f(t,0)\sin \pi t\, dt = 0$, Gupta [7] established the existence and uniqueness results of problem (1.1) with the Lidstone boundary conditions. Recently, in the case where F contains bending term u'' and under the particular boundary conditions, the authors of paper [9,12] have studied the existence of positive solutions for (1.1) when F satisfies the following growth condition:

$$\big|F(t,x,y) - (\alpha x - \beta y)\big| \leqslant a|x| + b|y| + c,$$

where $\alpha, \beta \in \mathbb{R}$ and $a, b, c > 0$, a, b is small enough. In addition, it is worth mentioning that Ma [5] showed the existence of positive solution for the following BVP

$$\begin{cases} u^{(4)}(t) - f(t,u,u'') = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $f \in C([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty))$ is superlinear or sublinear and Liu [13] improved the results of Ma [5]. We notice the above papers all require that F satisfies some growth condition or assumptions of monotonicity which are essential for the technique used. Moreover F has no singularity at $x = 0$ and/or $y = 0$.

The aim of this paper is to consider the existence of positive solutions for the more general Sturm–Liouville boundary value problem under weaker conditions when nonlinearity F contains second-order derivatives u''. Here we not only allow p, g have singularity at $t = 0, 1$, but also allow $F(t,x,y)$ has singularity at $x = 0$ and/or $y = 0$. As far as we know, there were fewer works to be done when F has singularity at $x = 0$ and/or $y = 0$. This paper attempts to fill part of this gap in the literature.

This paper is organized as follows. In Section 2, we firstly present some properties of Green's functions that are used to define a positive operator. Next we approximate the singular fourth-order boundary value problem to singular second-order boundary value problem by constructing an integral operator. Then in Section 3 the existence of positive solution for BVP (1.1) will be established by using the fixed point theory in cone, which we state here for convenience of the reader.

Let K be a cone in a Banach space E and let $K_r = \{x \in K\colon\ \|x\| < r\}$, $\partial K_r = \{x \in K\colon\ \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K\colon\ r \leqslant \|x\| \leqslant R\}$, where $0 < r < R < +\infty$.

Lemma 1.1. [16] *Let K be a positive cone in real Banach space E, $0 < r < R < +\infty$, and let $T : \overline{K}_{r,R} \to K$ be a completely continuous operator and such that*

(i) $\|Tx\| \leqslant \|x\|$ *for* $x \in \partial K_R$.
(ii) *There exists $e \in \partial K_1$ such that $x \neq Tx + me$ for any $x \in \partial K_r$ and $m > 0$.*

Then T has a fixed point in $\overline{K}_{r,R}$.

Remark 1.1. If (i) and (ii) are satisfied for $x \in \partial K_r$ and $x \in \partial K_R$, respectively, then Lemma 1.1 is still true.

2. Some preliminaries and lemmas

Definition 2.1. A function u is said to be a solution of the boundary value problem (1.1) if $u \in C^2([0,1], \mathbb{R}^+) \cap C^3((0,1), \mathbb{R}^+)$ satisfies $p(t)u'''(t) \in C^1((0,1), \mathbb{R}^+)$ and the BVP (1.1). In addition, u is said to be a positive solution if $u(t) > 0$ for $t \in (0,1)$ and u is solution of BVP (1.1). We notice that if $u(t)$ is a positive solution of the BVP (1.1) and $p \in C^1(0,1)$, then $u(t) \in C^{(4)}(0,1)$. For some λ, if the boundary value problem (1.1) has a positive solution u, then λ is called an eigenvalue and u is called a corresponding eigenfunction of the BVP (1.1).

Now we denote the Green's functions for the following boundary value problems

$$\begin{cases} -u'' = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \delta_1 u'(1) = 0, \end{cases}$$

and

$$\begin{cases} \dfrac{1}{p(t)}\big(p(t)v'(t)\big)' = 0, \quad 0 < t < 1, \\ \alpha_2 v(0) - \lim\limits_{t \to 0+} \beta_2 p(t) v'(t) = 0, \\ \gamma_2 v(1) + \lim\limits_{t \to 1-} \delta_2 p(t) v'(t) = 0, \end{cases}$$

by $H(t,s)$ and $G(t,s)$, respectively. It is well known that $H(t,s)$ and $G(t,s)$ can be written by

$$H(t,s) = \frac{1}{\rho_1} \begin{cases} (\beta_1 + \alpha_1 s)(\delta_1 + \gamma_1(1-t)), & 0 \leqslant s \leqslant t \leqslant 1, \\ (\beta_1 + \alpha_1 t)(\delta_1 + \gamma_1(1-s)), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.1}$$

and

$$G(t,s) = \frac{1}{\rho_2} \begin{cases} (\beta_2 + \alpha_2 B(0,s))(\delta_2 + \gamma_2 B(t,1)), & 0 \leqslant s \leqslant t \leqslant 1, \\ (\beta_2 + \alpha_2 B(0,t))(\delta_2 + \gamma_2 B(s,1)), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.2}$$

where $\rho_1 = \gamma_1\beta_1 + \alpha_1\gamma_1 + \alpha_1\delta_1$, $B(t,s) = \int_t^s \frac{d\tau}{p(\tau)}$, $\rho_2 = \alpha_2\delta_2 + \alpha_2\gamma_2 B(0,1) + \beta_2\gamma_2$. It is easy to verify the following properties of $G(t,s)$:

(I) $G(t,s) \leqslant G(s,s) \leqslant \frac{1}{\rho_2}(\beta_2 + \alpha_2 B(0,1))(\delta_2 + \gamma_2 B(0,1)) < +\infty$;
(II) $G(t,s) \geqslant \omega G(s,s)$, for any $t \in [a,b] \subset (0,1)$, $s \in [0,1]$, where

$$\omega = \min\left\{\frac{\delta_2 + \gamma_2 B(b,1)}{\delta_2 + \gamma_2 B(0,1)}, \frac{\beta_2 + \alpha_2 B(0,a)}{\beta_2 + \alpha_2 B(0,1)}\right\}. \tag{2.3}$$

Throughout this paper, we adopt the following assumptions:

(H_1) $p \in C^1((0,1),(0,+\infty))$, $0 < \int_0^1 \frac{ds}{p(s)} < +\infty$.
(H_2) $g \in C((0,1),[0,+\infty))$, $0 < \int_0^1 G(s,s)p(s)g(s)\,ds < +\infty$.
(H_3) $F(t,u,v) \in C((0,1)\times(0,+\infty)\times(-\infty,0),[0,+\infty))$, and for any $0 < r < R < +\infty$,

$$\lim_{n\to+\infty} \sup_{u,-v\in\overline{K}_{r,R}} \int_{D(n)} G(s,s)p(s)g(s)F\big(s,u(s),v(s)\big)\,ds = 0,$$

where $D(n) = [0,\frac{1}{n}] \cup [\frac{n-1}{n},1]$.

By (H_2), there exist $a,b \in (0,1)$ with $a < b$ such that

$$0 < \int_a^b G(s,s)p(s)g(s)\,ds < +\infty.$$

Consequently, by (II), we have

$$0 < \omega\int_a^b G(s,s)p(s)g(s)\,ds \leqslant \min_{t\in[a,b]}\int_0^1 G(t,s)p(s)g(s)\,ds$$
$$\leqslant \int_0^1 G(s,s)p(s)g(s)\,ds < +\infty.$$

So in the rest of this paper, a,b will be taken in this way, it is rather straightforward that

$$0 < \omega G(s,s) \leqslant G(t,s), \quad t \in [a,b] \subset (0,1),\ s \in [0,1],$$

and

$$0 < \min_{t\in[a,b]}\int_0^1 G(t,s)p(s)g(s)\,ds < +\infty.$$

Remark 2.1. For notational convenience, we introduce the following constants

$$\mu = \max_{0\leqslant t\leqslant 1}\int_0^1 H(t,s)\,ds,$$

and

$$L = (\mu+1)\int_0^1 G(s,s)p(s)g(s)\,ds, \qquad l = \min_{t\in[a,b]}\int_0^1 G(t,s)p(s)g(s)\,ds. \tag{2.4}$$

Obviously, $0 < l < L < +\infty$ and $0 < \mu \leqslant 1$.

Now we define an integral operator $S : C[0,1] \to C[0,1]$ by

$$Sv(t) = \int_0^1 H(t,\tau)v(\tau)\,d\tau.$$

Then by (2.1), we have

$$\begin{cases} (Sv)''(t) = -v(t), & 0 < t < 1, \\ \alpha_1 (Sv)(0) - \beta_1 (Sv)'(0) = 0, \\ \gamma_1 (Sv)(1) + \delta_1 (Sv)'(1) = 0. \end{cases} \tag{2.5}$$

Lemma 2.1. *The Sturm–Liouville boundary value problem* (1.1) *has a positive solution if and only if the following integral-differential boundary value problem*

$$\begin{cases} -\dfrac{1}{p(t)}\big(p(t)v'(t)\big)' = \lambda g(t) F\big(t, Sv(t), -v(t)\big), & 0 < t < 1, \\ \alpha_2 v(0) - \lim_{t\to 0+} \beta_2 p(t) v'(t) = 0, \\ \gamma_2 v(1) + \lim_{t \to 1-} \delta_2 p(t) v'(t) = 0, \end{cases} \tag{2.6}$$

has a positive solution.

Proof. In fact, if u is a positive solution of (1.1), let $u = Sv$, then $v = -u''$. This implies $u'' = -v$ is a solution of (2.6). Conversely, if v is a positive solution of (2.6), let $u = Sv$, by (2.5), $u'' = (Sv)'' = -v$. Thus $u = Sv$ is a positive solution of (1.1). This completes the proof of Lemma 2.1. $\square$

So we will concentrate our study on (2.6). Let $C^+[0,1] = \{x \in C[0,1]:\ x \geqslant 0\}$ and

$$K = \Big\{ x \in C^+[0,1]:\ x(t) \text{ is concave function on } [0,1],\ \min_{t\in[a,b]} x(t) \geqslant \omega \|x\| \Big\},$$

where ω is a constant defined by (2.3), $\|x\| = \sup_{t\in[0,1]} |x(t)|$, $x(t) \in C[0,1]$. It is easy to see that K is a cone in $C[0,1]$ and $\overline{K}_{r,R} \subset K \subset C^+[0,1]$. Now we define an operator $T : K\backslash\{0\} \to C^+[0,1]$ by

$$(Tv)(t) = \lambda \int_0^1 G(t,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds, \quad t \in [0,1].$$

Clearly v is a solution of the BVP (2.6) if and only if v is a fixed point of the operator T.

Lemma 2.2. *Assume that* (H_1)–(H_3) *hold. Then* $T : \overline{K}_{r,R} \to C^+[0,1]$ *is a completely continuous operator.*

Proof. Firstly, for any $r > 0$, we will show

$$\sup_{v\in\partial K_r} \lambda \int_0^1 G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds < +\infty. \tag{2.7}$$

At the same time, this implies $T: K\backslash\{0\} \to C^+[0,1]$ is well defined.

In fact, by (H_3), for any $r > 0$, there exists a natural number m such that

$$\sup_{v\in\partial K_r} \lambda \int\limits_{D(m)} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds < 1.$$

For any $v \in \partial K_r$, let $v(t_0) = \max_{t\in[0,1]} |v(t)| = r$. It follows from the concavity of $v(t)$ on [0,1] that

$$v(t) \geqslant \begin{cases} \frac{rt}{t_0}, & 0 \leqslant t \leqslant t_0, \\ \frac{r}{1-t_0}(1-t), & t_0 \leqslant t \leqslant 1. \end{cases}$$

So we obtain

$$v(t) \geqslant \begin{cases} rt, & 0 \leqslant t \leqslant t_0, \\ r(1-t), & t_0 \leqslant t \leqslant 1. \end{cases} \tag{2.8}$$

Consequently, from (2.8) for any $t \in [\frac{1}{m}, \frac{m-1}{m}]$, we have $\frac{r}{m} \leqslant v(t) \leqslant r$ and

$$\frac{l_m r}{m} = \frac{r}{m} \min_{t\in[\frac{1}{m},\frac{m-1}{m}]} \int_0^1 H(t,s)\,ds \leqslant Sv(t) \leqslant r \max_{t\in[\frac{1}{m},\frac{m-1}{m}]} \int_0^1 H(t,s)\,ds \leqslant \mu r,$$

where $l_m = \min_{t\in[\frac{1}{m},\frac{m-1}{m}]} \int_0^1 H(t,s)\,ds$ and μ is defined by (2.4). Let

$$M_1 = \max\left\{F(t,x,y)\colon\ (t,x,y) \in \left[\frac{1}{m}, \frac{m-1}{m}\right] \times \left[\frac{l_m r}{m}, \mu r\right] \times \left[-r, -\frac{r}{m}\right]\right\}.$$

By (H_1)–(H_3), we have

$$\begin{aligned} &\sup_{v\in\partial K_r} \lambda \int_0^1 G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\ &\quad\leqslant \sup_{v\in\partial K_r} \lambda \int\limits_{D(m)} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\ &\qquad + \sup_{v\in\partial K_r} \lambda \int_{\frac{1}{m}}^{\frac{m-1}{m}} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\ &\quad\leqslant 1 + M_1\lambda \int_0^1 G(s,s)p(s)g(s)\,ds < +\infty, \end{aligned} \tag{2.9}$$

i.e., (2.7) holds. This also implies $T(B)$ is uniformly bounded for any bounded set $B \subset \overline{K}_{r,R}$ from (2.9).

Next we prove T is equicontinuous on $\overline{K}_{r,R}$. In fact, by (H_3) for any $\varepsilon > 0$, there exists a natural number k such that

$$\sup_{v\in\overline{K}_{r,R}} \lambda \int\limits_{D(k)} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds < \frac{\varepsilon}{4}.$$

Let

$$M_2 = \max\left\{F(t,x,y)\text{: } (t,x,y) \in \left[\frac{1}{k}, \frac{k-1}{k}\right] \times \left[\frac{l_k r}{k}, \mu r\right] \times \left[-r, -\frac{r}{k}\right]\right\},$$

where $l_k = \min_{t\in[\frac{1}{k},\frac{k-1}{k}]} \int_0^1 H(t,s)\,ds$. Since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for the above $\varepsilon > 0$ and fixed $s \in [\frac{1}{k}, \frac{k-1}{k}]$, there exists $\delta > 0$ such that

$$\left|G(t,s) - G(t',s)\right| \leqslant (2\lambda L M_2)^{-1}(\mu+1)G(s,s)\varepsilon$$

for $|t-t'| < \delta$ and $t,t' \in [0,1]$. Consequently, when $|t-t'| < \delta$ and $t,t' \in [0,1]$, we have

$$\begin{aligned}\left|Tv(t) - Tv(t')\right| \leqslant 2 &\sup_{v\in\overline{K}_{r,R}} \lambda \int\limits_{D(k)} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\ &+ \lambda \sup_{v\in\overline{K}_{r,R}} \int\limits_{\frac{1}{k}}^{\frac{k-1}{k}} \left|G(t,s) - G(t',s)\right| p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\ &< \varepsilon.\end{aligned}$$

This implies that $T(\overline{K}_{r,R})$ is equicontinuous. Then by the Arzela–Ascoli theorem $T:\overline{K}_{r,R} \to C^+[0,1]$ is compact.

Finally, we show $T:\overline{K}_{r,R} \to C^+[0,1]$ is continuous. Assume $v_n, v_0 \in \overline{K}_{r,R}$ and $\|v_n - v_0\| \to 0$ $(n\to\infty)$. Then $r \leqslant \|v_n\| \leqslant R$ and $r \leqslant \|v_0\| \leqslant R$. For any $\varepsilon > 0$, by (H_3), there exists a natural number $m > 0$ such that

$$\sup_{v\in\overline{K}_{r,R}} \lambda \int\limits_{D(m)} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds < \frac{\varepsilon}{4}. \tag{2.10}$$

On the other hand, by (2.8), for any $t \in [\frac{1}{m}, \frac{m-1}{m}]$, we have

$$\frac{r}{m} \leqslant v_n(t) \leqslant R, \qquad \frac{l_m r}{m} \leqslant Sv_n(t) \leqslant \mu R, \qquad n = 0,1,2,\ldots,$$

where $l_m = \min_{t\in[\frac{1}{m},\frac{m-1}{m}]} \int_0^1 H(t,s)\,ds$ and μ is defined by (2.4).

Since $F(t,x,y)$ is uniformly continuous in $[\frac{1}{m}, \frac{m-1}{m}] \times [\frac{l_m r}{m}, \mu R] \times [-R, -\frac{r}{m}]$, we have

$$\lim_{n\to+\infty} \left|F\big(s, Sv_n(s), -v_n(s)\big) - F\big(s, Sv_0(s), -v_0(s)\big)\right| = 0$$

holds uniformly on $s \in [\frac{1}{m}, \frac{m-1}{m}]$. Then the Lebesgue dominated convergence theorem yields that

$$\int\limits_{\frac{1}{m}}^{\frac{m-1}{m}} \lambda G(s,s)p(s)g(s)\left|F\big(s, Sv_n(s), -v_n(s)\big) - F\big(s, Sv_0(s), -v_0(s)\big)\right|\,ds \to 0,$$

as $n \to \infty$.

Thus for above $\varepsilon > 0$, there exists a natural number N, for $n > N$, we have

$$\int\limits_{\frac{1}{m}}^{\frac{m-1}{m}} \lambda G(s,s)p(s)g(s)\left|F\big(s, Sv_n(s), -v_n(s)\big) - F\big(s, Sv_0(s), -v_0(s)\big)\right|\,ds < \frac{\varepsilon}{2}. \tag{2.11}$$

It follows from (2.10), (2.11) that when $n > N$,

$$\begin{aligned}\|Tv_n - Tv_0\| &\leqslant \int_{\frac{1}{m}}^{\frac{m-1}{m}} \lambda G(s,s)p(s)g(s)\bigl|F\bigl(s, Sv_n(s), -v_n(s)\bigr) - F\bigl(s, Sv_0(s), -v_0(s)\bigr)\bigr|\,ds \\ &\quad + 2\sup_{v\in\overline{K}_{r,R}} \int_{D(m)} \lambda G(s,s)p(s)g(s)F\bigl(s, Sv(s), -v(s)\bigr)\,ds \\ &< \frac{\varepsilon}{2} + 2\times\frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

This implies that $T:\overline{K}_{r,R} \to C^+[0,1]$ is continuous. Thus $T:\overline{K}_{r,R} \to C^+[0,1]$ is completely continuous. $\square$

Lemma 2.3. $T(\overline{K}_{r,R}) \subset K$.

Proof. For any $\overline{K}_{r,R}$, $t \in [0,1]$, we have

$$\begin{aligned}(Tv)(t) &= \lambda\int_0^1 G(t,s)p(s)g(s)F\bigl(s, Sv(s), -v(s)\bigr)\,ds \\ &\leqslant \lambda\int_0^1 G(s,s)p(s)g(s)F\bigl(s, Sv(s), -v(s)\bigr)\,ds.\end{aligned}$$

Thus

$$\|Tv\| \leqslant \lambda\int_0^1 G(s,s)p(s)g(s)F\bigl(s, Sv(s), -v(s)\bigr)\,ds.$$

On the other hand, by (II) we have

$$\begin{aligned}\min_{t\in[a,b]}(Tv)(t) &= \min_{t\in[a,b]}\lambda\int_0^1 G(t,s)p(s)g(s)F\bigl(s, Sv(s), -v(s)\bigr)\,ds \\ &\geqslant \lambda\omega\int_0^1 G(s,s)p(s)g(s)F\bigl(s, Sv(s), -v(s)\bigr)\,ds.\end{aligned}$$

This implies that $\min_{t\in[a,b]}(Tv)(t) \geqslant \omega\|Tv\|$. In addition, it is clear that Tv is concave on [0,1]. Thus $Tv \in K$. Consequently $T(\overline{K}_{r,R}) \subset K$. $\square$

3. Main results

Theorem 3.1. *Assume that conditions* (H_1)–(H_3) *are satisfied. Further assume that the following condition* (H_4) *holds*:

(H4) $$0 \leqslant F^0 = \limsup_{\substack{|x|+|y|\to 0\\ x>0,\ y<0}} \max_{t\in[0,1]} \frac{F(t,x,y)}{|x|+|y|} < L^{-1}$$

and

$$0 < l^{-1} < F_\infty = \liminf_{\substack{|x|+|y|\to +\infty\\ x>0,\ y<0}} \min_{t\in[a,b]} \frac{F(t,x,y)}{|x|+|y|} \leqslant +\infty.$$

Then the boundary value problem (1.1) *has at least one positive solution for any*

$$\lambda \in \left(\frac{1}{lF_\infty}, \frac{1}{LF^0}\right), \tag{3.1}$$

where L and l are defined by (2.4).

Proof. Let λ satisfy (3.1) and $\varepsilon > 0$ be chosen such that

$$F_\infty - \varepsilon > 0, \qquad \frac{1}{(F_\infty - \varepsilon)l} \leqslant \lambda \leqslant \frac{1}{(F^0+\varepsilon)L}. \tag{3.2}$$

Next, by (H4) there exists $r_0 > 0$ such that

$$F(t,x,y) \leqslant \big(F^0+\varepsilon\big)\big(|x|+|y|\big), \quad \forall t \in [0,1], \quad 0 < |x|+|y| < r_0,\ x>0,\ y<0. \tag{3.3}$$

Take $r = \frac{r_0}{\mu+1}$. Notice that

$$0 < |Sv| + |v| \leqslant (\mu+1)\|v\| = (\mu+1)r = r_0, \quad 0 \leqslant t \leqslant 1. \tag{3.4}$$

It follows from (3.3) and (3.4) that, for any $v \in \partial K_r$,

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\
&\leqslant \lambda \int_0^1 G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\
&\leqslant \lambda\big(F^0+\varepsilon\big) \int_0^1 G(s,s)p(s)g(s)\big(\big|Sv(s)\big| + \big|v(s)\big|\big)\,ds \\
&= \lambda\big(F^0+\varepsilon\big)(\mu+1)r \int_0^1 G(s,s)p(s)g(s)\,ds \\
&= \lambda L\big(F^0+\varepsilon\big)r \\
&\leqslant r = \|v\|.
\end{aligned}$$

Thus, $\|Tv\| \leqslant \|v\|$, $\forall v \in \partial K_r$.

On the other hand, for the above ε, by (H4), there exists $R_0 > 0$ such that

$$F(t,x,y) > (F_\infty - \varepsilon)\big(|x|+|y|\big), \quad t \in [a,b],\ |x|+|y| \geqslant R_0,\ x>0,\ y<0.$$

Let $R = \max\{2r, \omega^{-1}R_0\}$ and $\varphi(t) \equiv 1$, $t \in [0,1]$. Then $R > r$ and $\varphi(t) \in \partial K_1$.

In the following we show $v \neq Tv + m\varphi$ $(m > 0)$. Otherwise, there exist $v_0 \in \partial K_R$ and $m_0 > 0$ such that $v_0 = Tv_0 + m_0\varphi$. Let $\xi = \min\{v_0(t)\colon t \in [a,b]\}$ and notice that for any $s \in [a,b]$,

$$\left|Sv_0(s)\right| + \left|v_0(s)\right| \geqslant \min_{s\in[a,b]}\left[\left|Sv_0(s)\right| + \left|v_0(s)\right|\right] \geqslant \min_{s\in[a,b]}\left|v_0(s)\right| \geqslant \omega\|v_0\| \geqslant R_0.$$

Consequently for any $t \in [a,b]$, we have

$$\begin{aligned}
v_0(t) &= \lambda \int_0^1 G(t,s)p(s)g(s)F\big(s, Sv_0(s), -v_0(s)\big)\,ds + m_0\varphi(t) \\
&\geqslant \lambda \int_a^b G(t,s)p(s)g(s)F\big(s, Sv_0(s), -v_0(s)\big)\,ds + m_0 \\
&\geqslant \lambda(F_\infty - \varepsilon) \int_a^b G(t,s)p(s)g(s)\big(\left|Sv_0(s)\right| + \left|v_0(s)\right|\big)\,ds + m_0 \\
&\geqslant \lambda(F_\infty - \varepsilon) \int_a^b G(t,s)p(s)g(s)v_0(s)\,ds + m_0 \\
&\geqslant \lambda(F_\infty - \varepsilon)\xi \min_{t\in[a,b]} \int_a^b G(t,s)p(s)g(s)\,ds + m_0 \\
&\geqslant \xi + m_0 > \xi.
\end{aligned}$$

This implies that $\xi > \xi$ which is a contradiction. It follows from Lemma 1.1 that T has a fixed point v^* with $r < |v^*| < R$. Thus v^* is a positive solution of the BVP (2.6). Consequently, by Lemma 2.1, one can obtain that BVP (1.1) has a positive solution. This completes the proof of Theorem 3.1. $\square$

Remark 3.1. Since $0 < l < L < +\infty$, we easily obtain $0 < \frac{1}{lF_\infty} < 1$, $\frac{1}{LF^0} > 1$. Thus $1 \in (\frac{1}{lF_\infty}, \frac{1}{LF^0})$, so when $\lambda = 1$, Theorem 3.1 always holds.

Remark 3.2. From Theorem 3.1, we can see $F(t,x,y)$ need not be superlinear or sublinear. In fact, Theorem 3.1 still holds, if one of the following conditions is satisfied:

(i) if $F_\infty = \infty$, $F^0 > 0$, then for each $\lambda \in (0, \frac{1}{LF^0})$;
(ii) if $F_\infty = \infty$, $F^0 = 0$, then for each $\lambda \in (0, +\infty)$;
(iii) if $F_\infty > l^{-1} > 0$, $F^0 = 0$, then for each $\lambda \in (\frac{1}{lF_\infty}, +\infty)$.

Theorem 3.2. *Assume that conditions* (H_1)–(H_3) *are satisfied. Further assume that the following condition* (H_5) *holds*:

(H_5) $$0 \leqslant F^\infty = \limsup_{\substack{|x|+|y|\to+\infty \\ x>0,\ y<0}} \max_{t\in[0,1]} \frac{F(t,x,y)}{|x|+|y|} < L^{-1}$$

and

$$0 < l^{-1} < F_0 = \liminf_{\substack{|x|+|y|\to 0\\ x>0,\ y<0}} \min_{t\in[a,b]} \frac{F(t,x,y)}{|x|+|y|} \leqslant +\infty.$$

Then the boundary value problem (1.1) *has at least one positive solution for any*

$$\lambda \in \left(\frac{1}{lF_0}, \frac{1}{LF^\infty}\right), \tag{3.5}$$

where L and l are defined by (2.4).

Proof. Let λ satisfy (3.5) and let $\varepsilon_1 > 0$ be chosen such that $L^{-1} - \varepsilon_1 > 0$ and $\lambda F^\infty < L^{-1} - \varepsilon_1$. By (H_5), there exists $\mu R_0'$ such that

$$F(t,x,y) \leqslant \frac{1}{\lambda}\big(L^{-1} - \varepsilon_1\big)\big(|x| + |y|\big), \quad |x| + |y| \geqslant \mu R_0', \quad x > 0,\ y < 0,\ t \in [0,1].$$

Let

$$M_0 = \sup_{v\in\partial K_{R_0'}} \lambda \int_0^1 G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds.$$

Then $M_0 < +\infty$ by (2.7). Take $R_1 > \max\{R_0', \frac{M_0}{L\varepsilon_1}\}$, then $M_0 < LR_1\varepsilon_1$.

Notice $u \in \partial K_{R_0'}$ implies that

$$u(t) \leqslant \|u\| = R_0', \qquad Su(t) \leqslant \max_{t\in[0,1]} \int_0^1 H(t,s)\,ds\,\|u\| = \mu R_0'.$$

So for any $v \in \partial K_{R_1}$, let

$$D(Sv, -v) = \big\{t \in [0,1]\colon\ (Sv, -v) \in \big[\mu R_0', +\infty\big) \times \big(-\infty, -R_0'\big]\big\},$$

then for any $t \in D(Sv, -v)$, clearly $\mu R_0' \leqslant |Sv| + |v| \leqslant (\mu + 1)\|v\| = (\mu + 1)R_1$.

In addition, for any $v \in \partial K_{R_1}$, let $v_1(t) = \min\{v(t), R_0'\}$, then $v_1 \in \partial K_{R_0'}$. Thus, for any $v \in \partial K_{R_1}$, we have

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\
&\leqslant \max_{t\in[0,1]} \lambda \int_{D(Sv,-v)} G(t,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\
&\quad + \lambda \int_{[0,1]/D(Sv,-v)} G(s,s)p(s)g(s)F\big(s, Sv(s), -v(s)\big)\,ds \\
&\leqslant \frac{1}{\lambda}\big(L^{-1} - \varepsilon_1\big)\lambda \int_0^1 G(s,s)p(s)g(s)\big(\big|Sv(s)\big| + \big|v(s)\big|\big)\,ds
\end{aligned}$$

$$+\lambda\int_0^1 G(s,s)p(s)g(s)F\big(s,Sv_1(s),-v_1(s)\big)\,ds$$
$$\leqslant \big(L^{-1}-\varepsilon_1\big)(\mu+1)R_1\int_0^1 G(s,s)p(s)g(s)\,ds+M_0$$
$$\leqslant \big(L^{-1}-\varepsilon_1\big)LR_1+M_0$$
$$< R_1=\|v\|.$$

Therefore $\|Tv\|\leqslant\|v\|$ for any $v\in\partial K_{R_1}$.

Next, let λ satisfy (3.5). Choose $\varepsilon_2>0$ such that $l^{-1}+\varepsilon_2<\lambda F_0$. Then from ($H_5$), there exists $0<\delta<(\mu+1)R_1$ such that

$$F(t,x,y)\geqslant\frac{1}{\lambda}\big(l^{-1}+\varepsilon_2\big)\big(|x|+|y|\big),\quad 0<|x|+|y|\leqslant\delta,\quad x>0,\ y<0,\ t\in[a,b].$$

Let $r_1=\frac{\delta}{\mu+1}$ and $\phi(t)\equiv 1$, $t\in[0,1]$, then $r_1<R_1$ and $\phi\in\partial K_1$.

Now we prove $v\neq Tv+m\phi$ ($m>0$). Otherwise, there exist $v_0\in\partial K_{r_1}$ and $m_0>0$ such that $v_0=Tv_0+m_0\phi$. Let $\zeta=\min\{v_0(t)\colon t\in[a,b]\}$ and apply that $|v_0(s)|+|Sv_0(s)|<(\mu+1)r_1=\delta$, then for any $t\in[a,b]$, we have

$$v_0(t)=\lambda\int_0^1 G(t,s)p(s)g(s)F\big(s,Sv_0(s),-v_0(s)\big)\,ds+m_0\phi(t)$$
$$\geqslant\lambda\int_a^b G(t,s)p(s)g(s)F\big(s,Sv_0(s),-v_0(s)\big)\,ds+m_0$$
$$\geqslant\frac{1}{\lambda}\big(l^{-1}+\varepsilon_2\big)\lambda\int_a^b G(t,s)p(s)g(s)\big(\big|Sv_0(s)\big|+\big|v_0(s)\big|\big)\,ds+m_0$$
$$\geqslant\big(l^{-1}+\varepsilon_2\big)\int_a^b G(t,s)p(s)g(s)v_0(s)\,ds+m_0$$
$$\geqslant\big(l^{-1}+\varepsilon_2\big)\zeta\min_{t\in[a,b]}\int_a^b G(t,s)p(s)g(s)\,ds+m_0$$
$$\geqslant(1+l\varepsilon_2)\zeta+m_0>\zeta.$$

This implies that $\zeta>\zeta$ which is a contradiction. It follows from Lemma 1.1 that T has a fixed point v^{**} with $r_1<|v^{**}|<R_1$. Thus v^{**} is a positive solution of the BVP (2.6). By Lemma 2.1, BVP (1.1) has a positive solution. □

Remark 3.3. From Theorem 3.2 we can see the conclusions still hold, if one of the following conditions is satisfied:

(i) if $F^\infty<L^{-1}$, $F_0=\infty$, then for each $\lambda\in(0,\frac{1}{LF^\infty})$;

(ii) if $F^{\infty}=0$, $F_0=+\infty$, then for each $\lambda\in(0,+\infty)$;
(iii) if $F^{\infty}=0$, $F_0>l^{-1}>0$, then for each $\lambda\in(\frac{1}{lF_0},+\infty)$.

Remark 3.4. Note that if F is superlinear (i.e., $F^0=0$, $F_{\infty}=+\infty$) or sublinear (i.e., $F_0=+\infty$, $F^{\infty}=0$), BVP (1.1) has at least one positive solution for any $\lambda\in(0,+\infty)$. In particular, if $p(t)=1$ and $g(t)F(t,u,u'')=a(t)f(u'')$, or $\lambda=1$, the conclusions of Theorems 3.1 and 3.2 hold. So our conclusions extend and improve the corresponding results of papers [5,13].

Remark 3.5. Because of singularity of p, g, F, it seems to be difficult to prove our results by using the norm-type expansion and compression theorem as was done in [5,8–10,13]. In addition, it needs to be pointed out that we not only obtain the existence of positive solutions of BVP (1.1), but also get the explicit interval about λ, which is different from the previous papers (see [3,5,6,15]).

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